

INVARIANT MEASURES AND MAXIMAL L^2 REGULARITY FOR NONAUTONOMOUS ORNSTEIN-UHLENBECK EQUATIONS

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ABSTRACT. We characterize the domain of the realizations of the linear parabolic operator \mathcal{G} defined by (1.4) in L^2 spaces with respect to a suitable measure, that is invariant for the associated evolution semigroup. As a byproduct, we obtain optimal L^2 regularity results for evolution equations with time-dependent Ornstein-Uhlenbeck operators.

1. INTRODUCTION

Finite dimensional Ornstein-Uhlenbeck operators are elliptic (possibly degenerate) differential operators of the type

$$\mathcal{L}\varphi(x) = \frac{1}{2}\text{Tr} (BB^*D^2\varphi(x)) + \langle Ax, D\varphi(x) \rangle, \quad x \in \mathbb{R}^n,$$

where A, B are given nonzero matrices. They are prototypes of Kolmogorov operators, and have been the object of several studies from the probabilistic and the deterministic point of view. The theory of linear elliptic and parabolic equations involving an Ornstein-Uhlenbeck operator, such as

$$(1.1) \quad \lambda\varphi(x) - \mathcal{L}\varphi(x) = f(x), \quad x \in \mathbb{R}^n,$$

$$(1.2) \quad \begin{cases} u_t(t, x) + \mathcal{L}u(t, x) = g(t, x), & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^n, \end{cases}$$

is now well developed, both in spaces of continuous bounded functions, and in L^p spaces. See e.g. [DPZ92, Lun97a, CG01, DPZ02, MPP02, MPRS02, FL06]. The most natural L^p setting for Ornstein-Uhlenbeck operators (as, more generally, for Kolmogorov operators) are not the usual L^p spaces with respect to the Lebesgue measure, but L^p spaces with respect to an invariant measure ζ , that is a Borel probability measure in \mathbb{R}^n satisfying

$$(1.3) \quad \int_{\mathbb{R}^n} T(t)\varphi \, d\zeta = \int_{\mathbb{R}^n} \varphi \, d\zeta, \quad t > 0, \varphi \in C_b(\mathbb{R}^n),$$

where $T(t)$ is the Ornstein-Uhlenbeck semigroup. Indeed, invariant measures arise naturally in the study of the asymptotic behavior of $T(t)$, and the realizations of \mathcal{L} in L^p spaces with respect to invariant measures are dissipative, and therefore they enjoy nice analytic properties.

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The aim of this paper is to extend a part of the theory to nonautonomous problems with time depending Ornstein-Uhlenbeck operators. Precisely, we consider the parabolic operator in $H_{\text{loc}}^{1,2}(\mathbb{R}^{1+n})$

$$(1.4) \quad \mathcal{G}u(t, x) = \partial_t u(t, x) + \mathcal{L}(t)u(t, \cdot)(x), \quad t \in \mathbb{R}, x \in \mathbb{R}^n,$$

where $\mathcal{L}(t)$ are given (on $H_{\text{loc}}^2(\mathbb{R}^n)$) by

$$(1.5) \quad \mathcal{L}(t)\varphi(x) = \frac{1}{2}\text{Tr} (B(t)B^*(t)D_x^2\varphi(x)) + \langle A(t)x + f(t), D_x\varphi(x) \rangle, \quad x \in \mathbb{R}^n,$$

with continuous and bounded data $A, B : \mathbb{R} \rightarrow \mathcal{L}(\mathbb{R}^n)$ and $f : \mathbb{R} \rightarrow \mathbb{R}^n$.

As in the autonomous case, the operator \mathcal{G} arises from linear stochastic Cauchy problems in \mathbb{R}^n ,

$$(1.6) \quad \begin{cases} dX_t = (A(t)X_t + f(t))dt + B(t)dW(t), & t \geq s, \\ X_s = x, \end{cases}$$

where $W(t)$ is a standard n -dimensional Brownian motion and $s \in \mathbb{R}, x \in \mathbb{R}^n$. Indeed, it is well known that, denoting by $X(s, t, x)$ the solution to (1.6), for each $t \in \mathbb{R}$ and $\varphi \in C_b^2(\mathbb{R}^n)$ the function $u(s, x) := \mathbb{E}(\varphi(X(s, t, x)))$ satisfies the backward Kolmogorov Cauchy problem

$$(1.7) \quad \begin{cases} u_s(s, x) + \mathcal{L}(s)u(s, x) = 0, & s \leq t, x \in \mathbb{R}^n, \\ u(t) = \varphi(x), & x \in \mathbb{R}^n. \end{cases}$$

See e.g. [GS72, KS91]. However, our approach is purely deterministic and it relies on the study of the backward evolution operator $P_{s,t}$ for (1.7) and of the associated evolution semigroup.

Throughout the paper we assume that \mathcal{G} is uniformly parabolic, i.e. there exists $\mu_0 > 0$ such that

$$(1.8) \quad \|B(t)x\| \geq \mu_0\|x\|, \quad t \in \mathbb{R}, x \in \mathbb{R}^n.$$

Moreover, we assume that the evolution family $U(t, s)$ generated by $A(\cdot)$ is stable, and we denote by $\omega_0(U)$ its growth bound. In other words,

$$(1.9) \quad \omega_0(U) := \inf\{ \omega \in \mathbb{R} : \exists M = M(\omega) \text{ such that } \|U(t, s)\| \leq Me^{\omega(t-s)}, \quad -\infty < s \leq t < \infty \} < 0.$$

While in the autonomous case these assumptions imply existence and uniqueness of a probability measure ζ satisfying (1.3), in our nonautonomous case there does not exist a unique ζ such that

$$\int_{\mathbb{R}^n} P_{s,t}\varphi d\zeta = \int_{\mathbb{R}^n} \varphi d\zeta, \quad s < t, \varphi \in C_b(\mathbb{R}^n),$$

but we can find families of measures $\{\mu_t : t \in \mathbb{R}\}$, called *entrance laws at time $-\infty$* in [Dyn89] and *evolution systems of measures* in [DPR05], such that

$$(1.10) \quad \int_{\mathbb{R}^n} P_{s,t}\varphi d\mu_s = \int_{\mathbb{R}^n} \varphi d\mu_t, \quad \varphi \in C_b(\mathbb{R}^n), s \leq t.$$

Such families of measures are infinitely many, and they are characterized in Section 2. Among all of them, the simplest one consists of the Gaussian measures ν_t defined by

$$(1.11) \quad \nu_t = \mathcal{N}_{g(t, -\infty), Q(t, -\infty)}, \quad t \in \mathbb{R},$$

where

$$g(t, s) := \int_s^t U(t, r) f(r) dr, \quad Q(t, s) := \int_s^t U(t, r) B(r) B^*(r) U^*(t, r) dr, \quad -\infty \leq s < t,$$

and we denote by $\mathcal{N}_{m, Q}$ the n -dimensional Gaussian measure with covariance operator Q and mean m .

With the aid of the measures ν_t , we construct a Borel measure in \mathbb{R}^{1+n} as follows: for $I \in \mathcal{B}(\mathbb{R})$ and $K \in \mathcal{B}(\mathbb{R}^n)$, we set $\nu(I \times K) = \int_I \nu_t(K) dt$, then ν is extended in a standard way to a measure on $\mathcal{B}(\mathbb{R}^{1+n})$, still denoted by ν .

We define $G_0 : D(G_0) \subset L^2(\mathbb{R}^{1+n}, \nu) \rightarrow L^2(\mathbb{R}^{1+n}, \nu)$ by

$$(G_0 u)(t, x) = \mathcal{G}u(t, x), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad u \in D(G_0),$$

where $D(G_0)$ is a core of nice functions; precisely, it is the linear span of real and imaginary parts of the functions u of the type $u(t, x) = \Phi_j(t) e^{i\langle x, h_j(t) \rangle}$ with $\Phi_j \in C_c^1(\mathbb{R})$ and $h_j \in C_b^1(\mathbb{R}; \mathbb{R}^n)$. Then ν is invariant for G_0 , in the sense that

$$(1.12) \quad \int_{\mathbb{R} \times \mathbb{R}^n} G_0 u(t, x) d\nu = 0, \quad u \in D(G_0).$$

A fundamental property of the realizations of second order elliptic and parabolic operators in L^2 spaces with respect to invariant measures is their dissipativity. In fact, since $G_0(u^2) = 2u G_0 u + |B^* D_x u|^2$ and the integral of $G_0(u^2)$ vanishes, we get

$$\int_{\mathbb{R}^{1+n}} u G_0 u d\nu = -\frac{1}{2} \int_{\mathbb{R}^{1+n}} |B^* D_x u|^2 d\nu \leq 0,$$

so that $\langle u, G_0 u \rangle_{L^2} \leq 0$ for each $u \in D(G_0)$. Being dissipative, G_0 is closable. Its closure G is dissipative and has dense domain because $D(G_0)$ is dense. G is the natural realization of \mathcal{G} in $L^2(\mathbb{R}^{1+n}, \nu)$, as next theorem 1.1 states.

For $k, s \in \mathbb{N}$ we define the Sobolev spaces

$$H^{k,s}(\mathbb{R}^{1+n}, \nu) := \left\{ u \in H_{\text{loc}}^{k,s}(\mathbb{R}^{1+n}) : \partial_t^l u \in L^2(\mathbb{R}^{1+n}, \nu) \text{ for all } 0 < l < k, \right. \\ \left. D_x^\alpha u \in L^2(\mathbb{R}^{1+n}, \nu) \text{ for all } |\alpha| \leq s \right\}.$$

Our first main result reads as follows.

Theorem 1.1. *We have*

$$D(G) = H^{1,2}(\mathbb{R}^{1+n}, \nu) = \left\{ u \in H_{\text{loc}}^{1,2}(\mathbb{R}^{1+n}) \cap L^2(\mathbb{R}^{1+n}, \nu) : \mathcal{G}u \in L^2(\mathbb{R}^{1+n}, \nu) \right\}.$$

Note that ν is not a probability measure, because of the Lebesgue measure with respect to time in the whole \mathbb{R} . To avoid this drawback we may work in spaces of time periodic functions, assuming that also the coefficients A, B, f are periodic with the same period T . Then $\frac{1}{T}\nu$ is a probability measure in $(0, T) \times \mathbb{R}^n$.

To be precise, let $L_{\#}^2(\mathbb{R}^{1+n})$ denote the Hilbert space of all Lebesgue measurable functions $u : \mathbb{R}^{1+n} \rightarrow \mathbb{R}$ such that $u(t, x) = u(t + T, x)$ a.e. $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $u|_{(0,T) \times \mathbb{R}^n} \in L^2((0, T) \times \mathbb{R}^n, \nu)$, endowed with the norm

$$u \mapsto \left(\frac{1}{T} \int_0^T \int_{\mathbb{R}^n} u(t, x)^2 d\nu_t dt \right)^{1/2}.$$

Similarly as above, and as in [DPL06], we define $G_0^{\#} : D(G_0^{\#}) \subset L_{\#}^2(\mathbb{R}^{1+n}, \nu) \mapsto L_{\#}^2(\mathbb{R}^{1+n}, \nu)$ where $D(G_0^{\#})$ is the linear span of real and imaginary parts of the functions u of the type $u(t, x) = \Phi_j(t) e^{i\langle x, h_j(t) \rangle}$ with T -periodic $\Phi_j \in C^1(\mathbb{R})$ and $h_j \in C^1(\mathbb{R}; \mathbb{R}^n)$. Again, $G_0^{\#}$ is dissipative, hence closable, and its closure $G^{\#}$ generates a C_0 -semigroup $(\mathcal{P}_{\tau}^{\#})_{\tau \geq 0}$ of contractions on $L_{\#}^2(\mathbb{R}^{1+n}, \nu)$.

For $k, s \in \mathbb{N}$ we set

$$\begin{aligned} H_{\#}^{k,s}(\mathbb{R}^{1+n}, \nu) &:= \{u \in H_{\text{loc}}^{k,s}(\mathbb{R}^{1+n}) : \partial_t^l u \in L_{\#}^2(\mathbb{R}^{1+n}, \nu) \text{ for all } 0 < l < k, \\ &\quad D_x^{\alpha} u \in L_{\#}^2(\mathbb{R}^{1+n}, \nu) \text{ for all } |\alpha| \leq s\}. \end{aligned}$$

The description of the domain in the T -periodic case reads as follows.

Theorem 1.2. *We have*

$$\begin{aligned} D(G_{\#}) &= H_{\#}^{1,2}(\mathbb{R}^{1+n}, \nu) \\ &= \left\{ u \in H_{\text{loc}}^{1,2}(\mathbb{R}^{1+n}) \cap L_{\#}^2(\mathbb{R}^{1+n}, \nu) : \mathcal{G}u \in L_{\#}^2(\mathbb{R}^{1+n}, \nu) \right\}. \end{aligned}$$

This characterization yields that $D(G_{\#})$ is compactly embedded in $L_{\#}^2(\mathbb{R}^{1+n}, \nu)$, through the compactness of the embedding $H_{\#}^{1,2}(\mathbb{R}^{1+n}, \nu) \subset L_{\#}^2(\mathbb{R}^{1+n}, \nu)$ that we prove in Section 5.

Theorems 1.1 and 1.2 may be seen as maximal regularity results for evolution equations with time in \mathbb{R} ,

$$(1.13) \quad u_t(t, x) + \mathcal{L}(t)u(t, x) = h(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

with datum $h \in L^2(\mathbb{R}^{1+n}, \nu)$ (respectively, $h \in L_{\#}^2(\mathbb{R}^{1+n}, \nu)$ in the periodic case), since they state that if $u \in L^2(\mathbb{R}^{1+n}, \nu) \cap H_{\text{loc}}^{1,2}(\mathbb{R}^{1+n}, \nu)$ (resp., $u \in L_{\#}^2(\mathbb{R}^{1+n}, \nu) \cap H_{\text{loc}}^{1,2}(\mathbb{R}^{1+n}, \nu)$) satisfies (1.13) then u_t and each second order space derivative $D_{ij}u$ belong to $L^2(\mathbb{R}^{1+n}, \nu)$ (respectively, to $L_{\#}^2(\mathbb{R}^{1+n}, \nu)$).

Concerning solvability of problem (1.13), we remark that it is not a Cauchy problem and we do not expect existence and uniqueness of a solution u for any h ; in fact, it is not hard to see that 0 is in the spectrum of G and of $G_{\#}$. (The spectral properties of G and of $G_{\#}$, as well as asymptotic behavior of $P_{s,t}$, will be studied in a forthcoming paper [GL07]).

Note that problem (1.13) cannot be seen as an evolution equation of the type $u'(t) + L(t)u(t) = h(t)$ in a fixed Hilbert space H , because the Hilbert spaces $L^2(\mathbb{R}^n, \nu_t)$ involved here vary with time. So we cannot use the techniques of evolution equations in (fixed) Hilbert spaces.

Our procedure is the following: we use the fact that G and $G_{\#}$ are the infinitesimal generators of the evolution semigroup

$$(1.14) \quad (\mathcal{P}_{\tau}u)(t, x) = (P_{t,t+\tau}u(t+\tau, \cdot))(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad \tau \geq 0,$$

in the spaces $L^2(\mathbb{R}^{1+n}, \nu)$ and $L^2_{\#}(\mathbb{R}^{1+n}, \nu)$, respectively. We prove optimal blow-up estimates for the space derivatives of $P_{s,t}\varphi$ near $t = s$ for any $\varphi \in L^2(\mathbb{R}^n, \nu_t)$ and any multi-index α ,

$$(1.15) \quad \|D_x^\alpha P_{s,t}\|_{\mathcal{L}(L^2(\mathbb{R}^n, \nu_t), L^2(\mathbb{R}^n, \nu_s))} \leq C(t-s)^{-|\alpha|/2}, \quad 0 < t-s < 1,$$

that yield optimal estimates for the norm of \mathcal{P}_τ in $\mathcal{L}(L^2(\mathbb{R}^{1+n}, \nu), H^{0,k}(\mathbb{R}^{1+n}, \nu))$ and in $\mathcal{L}(L^2_{\#}(\mathbb{R}^{1+n}, \nu), H^{0,k}_{\#}(\mathbb{R}^{1+n}, \nu))$ for $k \in \mathbb{N}$, near $\tau = 0$. Then we use an interpolation theorem that gives optimal embeddings for the domain of the infinitesimal generator of a semigroup $T(\tau)$ from optimal estimates on the behavior of $T(\tau)$ near $\tau = 0$; in our case it gives $D(L) \subset (L^2(\mathbb{R}^{1+n}, \nu), H^{0,4}(\mathbb{R}^{1+n}, \nu))_{1/2,2}$. The latter space is readily characterized as $H^{0,2}(\mathbb{R}^{1+n}, \nu)$.

The crucial step are the smoothing estimates (1.15) that are quite similar to the corresponding estimates in the autonomous case. Together with (1.15) we obtain also optimal estimates for $t-s \rightarrow \infty$,

$$(1.16) \quad \|D_x^\alpha P_{s,t}\|_{\mathcal{L}(L^2(\mathbb{R}^n, \nu_t), L^2(\mathbb{R}^n, \nu_s))} \leq C e^{\omega|\alpha|(t-s)}, \quad t-s > 1,$$

where ω is any number in $(\omega_0(U), 0)$ and $C = C(\alpha, \omega)$. These estimates will be the starting point of the study of spectral properties and asymptotic behavior of the forthcoming paper [GL07].

The characterization of the domain of G in Theorem 1.1 allows us to study maximal L^2 regularity in backward Cauchy problems such as

$$(1.17) \quad \begin{cases} u_s(s, x) + \mathcal{L}(s)u(s, x) = h(s), & s \in (T_1, T_2), x \in \mathbb{R}^n, \\ u(T_2, x) = \varphi(x), & x \in \mathbb{R}^n. \end{cases}$$

with fixed $T_1 < T_2$. Note that we do not assume the coefficients A , B and f to be $(T_2 - T_1)$ -periodic.

Theorem 1.3. *Let $T_1 < T_2$. For each $h \in L^2((T_1, T_2) \times \mathbb{R}^n, \nu)$ and $\varphi \in H^1(\mathbb{R}^n, \nu_{T_2})$ there exists a unique solution $u \in H^{1,2}((T_1, T_2) \times \mathbb{R}^n, \nu)$ of (1.17). Moreover, u satisfies*

$$(1.18) \quad \|u\|_{H^{1,2}((T_1, T_2) \times \mathbb{R}^n, \nu)} \leq C \left(\|h\|_{L^2((T_1, T_2) \times \mathbb{R}^n, \nu)} + \|\varphi\|_{H^1(\mathbb{R}^n, \nu_{T_2})} \right),$$

where $C > 0$ is independent of h and φ .

The assumption $u_0 \in H^1(\mathbb{R}^n, \nu_{T_2})$ is necessary for $u \in H^{1,2}((T_1, T_2) \times \mathbb{R}^n, \nu)$ because $H^1(\mathbb{R}^n, \nu_{T_2})$ is the space of the traces $u(T_2, \cdot)$ of the functions $u \in H^{1,2}((T_1, T_2) \times \mathbb{R}^n, \nu)$.

This paper is organized as follows. In section 2 we characterize all the families of probability measures $\{\mu_t : t \in \mathbb{R}\}$ such that

$$\int_{\mathbb{R}^n} P_{s,t}\varphi \, d\mu_s = \int_{\mathbb{R}^n} \varphi \, d\mu_t, \quad \varphi \in C_b(\mathbb{R}^n), \quad s \leq t,$$

and we show that the measures ν_t defined in (1.11) are the unique ones satisfying

$$\sup_{t \in \mathbb{R}} \int_{\mathbb{R}^n} |x|^\alpha \mu_t(dx) < +\infty$$

for some $\alpha > 0$. The proofs of the domain characterizations are given in Sections 5.1 and 5.2, respectively. Estimates (1.15) and (1.16) are proved in Section 3. The characterization of real interpolation spaces between $L^2(\mathbb{R}^{1+n}, \nu)$ and $H^{0,k}(\mathbb{R}^{1+n}, \nu)$ and between

$L^2_{\#}(\mathbb{R}^{1+n}, \nu)$ and $H^{0,k}_{\#}(\mathbb{R}^{1+n}, \nu)$ is given in Section 4. Finally, the proof of Theorem 1.3 is given in Section 6.

2. PRELIMINARIES AND NOTATION, INVARIANT MEASURES

We recall some general facts about time-dependent Ornstein-Uhlenbeck operators and their invariant measures, already partly discussed in [DPL06].

First of all, for all $\varphi \in C_b^2(\mathbb{R}^n)$ and $t \in \mathbb{R}$ there exists a unique bounded solution $u \in C^{1,2}(\{(s, x) \in \mathbb{R}^{1+n} : s \leq t\})$ of the problem

$$(2.1) \quad \begin{cases} \partial_s u(s, x) + \mathcal{L}(s)u(s, x) = 0, & x \in \mathbb{R}^n, s < t, \\ u(t, x) = \varphi(x), & x \in \mathbb{R}^n. \end{cases}$$

The solution $u(s, x) := P_{s,t}\varphi(x)$ of (2.1) is given by the formula

$$(2.2) \quad P_{s,t}\varphi(x) = \int_{\mathbb{R}^n} \varphi(y + g(t, s)) \mathcal{N}_{U(t,s)x, Q(t,s)}(dy), \quad -\infty < s \leq t < \infty.$$

This has been shown in [DPL06] under periodicity assumptions on the coefficients, but the proof goes through as well in the general case. It is easy to see that under our non-degeneration assumption (1.8), for each $\varphi \in C_b(\mathbb{R}^n)$ the function $u(s, x) = P_{s,t}\varphi(x)$ is still the unique bounded classical solution to (2.1). The associated evolution semigroup in $C_b(\mathbb{R}^{1+n})$ is defined by

$$\mathcal{P}_\tau u(t, x) = P_{t, t+\tau} u(t + \tau, \cdot)(x), \quad \tau \geq 0, x \in \mathbb{R}^n, t \in \mathbb{R}.$$

Definition 2.1. A measure ν in \mathbb{R}^{1+n} is said to be invariant for \mathcal{P}_τ if

$$(2.3) \quad \int_{\mathbb{R}^{1+n}} \mathcal{P}_\tau u \, d\nu = \int_{\mathbb{R}^{1+n}} u \, d\nu, \quad \tau > 0, u \in C_b(\mathbb{R}^{1+n}) \cap L^1(\mathbb{R}^{1+n}, \nu).$$

An evolution system of measures for $P_{s,t}$ is a family of probability measures $(\nu_t)_{t \in \mathbb{R}}$ in \mathbb{R}^n such that

$$(2.4) \quad \int_{\mathbb{R}^n} P_{s,t}\varphi(x) \nu_s(dx) = \int_{\mathbb{R}^n} \varphi(x) \nu_t(dx), \quad \varphi \in C_b(\mathbb{R}^n).$$

If $P_{s,t}$ has an evolution system of measures $(\nu_t)_{t \in \mathbb{R}}$, then one can define an invariant measure for \mathcal{P}_τ , as follows. For Borel sets $I \subset \mathbb{R}$, $K \subset \mathbb{R}^n$ we define $\nu(I \times K) = \int_I \nu_t(K) \, dt$, then ν is extended in a standard way to all $\mathcal{B}(\mathbb{R}^{1+n})$. It is easy to see that μ is invariant for \mathcal{P}_τ .

It is well known (Khas'minskiĭ Theorem) that if a Markov semigroup is strong Feller and irreducible, then it has at most one invariant measure. Our semigroup \mathcal{P}_τ is not irreducible and it is not strong Feller because of the translation part, and it has in fact infinitely many invariant measures, as the next proposition shows. For its proof, we recall that the Fourier transform of a Gaussian measure $\mathcal{N}_{m,Q}$ is given by

$$(2.5) \quad \widehat{\mathcal{N}}_{m,Q}(h) = e^{i\langle m, h \rangle - \frac{1}{2} \langle Qh, h \rangle}, \quad h \in \mathbb{R}^n.$$

Proposition 2.2. Fixed any $t_0 \in \mathbb{R}$ and any Borel probability measure μ in \mathbb{R}^n , define a family of Borel probability measures μ_t by their Fourier transforms,

$$(2.6) \quad \widehat{\mu}_t(h) := \widehat{\mu}(U^*(t, t_0)h), \quad t \in \mathbb{R}.$$

Then the measures ν_t defined by

$$(2.7) \quad \nu_t = \mathcal{N}_{g(t, -\infty), Q(t, -\infty)} \star \mu_t, \quad t \in \mathbb{R},$$

are an evolution system of measures for $P_{s,t}$. Moreover, all the evolution systems of measures for $P_{s,t}$ are of this type.

Proof. We remark that a family of Borel probability measures $(\nu_t)_{t \in \mathbb{R}}$ is an evolution system of measures iff their Fourier transforms satisfy

$$(2.8) \quad e^{i\langle g(t,s), h \rangle} e^{-\frac{1}{2}\langle Q(t,s)h, h \rangle} \widehat{\nu}_s(U^*(t, s)h) = \widehat{\nu}_t(h), \quad s \leq t, \quad h \in \mathbb{R}^n.$$

Indeed, the left hand side of (2.8) is equal to $\int_{\mathbb{R}^n} P_{s,t} \varphi(x) \nu_s(dx)$ and the right hand side is $\int_{\mathbb{R}^n} \varphi(x) \nu_t(dx)$ if we take $\varphi(x) := e^{i\langle x, h \rangle}$. So, if $(\nu_t)_{t \in \mathbb{R}}$ is an evolution system of measures the left and the right hand side have to coincide. Conversely, if φ is any continuous bounded function, there exists a sequence $(\varphi_k)_{k \in \mathbb{N}}$, where $\varphi_k \in \text{span}\{e^{i\langle x, h \rangle} : h \in \mathbb{R}^n\}$ such that $\|\varphi_k\|_\infty \leq \|\varphi\|_\infty$, and $\lim_{k \rightarrow \infty} \varphi_k(x) = \varphi(x)$, $\forall x \in \mathbb{R}^n$. By (2.8), the equality $\int_{\mathbb{R}^n} P_{s,t} \varphi_k(x) \nu_s(dx) = \int_{\mathbb{R}^n} \varphi_k(x) \nu_t(dx)$ holds for each k , and letting $k \rightarrow \infty$ we get (2.4).

If ν_t is defined by (2.7), then for each $h \in \mathbb{R}^n$ and $s < t$ we have

$$\begin{aligned} & e^{i\langle g(t,s), h \rangle - \frac{1}{2}\langle Q(t,s)h, h \rangle} \widehat{\nu}_s(U^*(t, s)h) \\ &= e^{i\langle g(t,s), h \rangle - \frac{1}{2}\langle Q(t,s)h, h \rangle} \widehat{\mathcal{N}}_{g(s, -\infty), Q(s, -\infty)}(U^*(t, s)h) \widehat{\mu}_s(U^*(t, s)h) \\ &= e^{i\langle g(t,s), h \rangle - \frac{1}{2}\langle Q(t,s)h, h \rangle} e^{i\langle g(s, -\infty), U^*(t,s)h \rangle - \frac{1}{2}\langle Q(s, -\infty)U^*(t,s)h, U^*(t,s)h \rangle} \widehat{\mu}(U^*(s, t_0)U^*(t, s)h) \\ &= e^{i\langle g(t, -\infty), h \rangle - \frac{1}{2}\langle Q(t, -\infty)h, h \rangle} \widehat{\mu}(U^*(t, t_0)h) \\ &= \widehat{\nu}_t(h), \end{aligned}$$

so that (2.8) holds.

Conversely, if (2.8) holds, then the left hand side is independent of s , hence for each $t \in \mathbb{R}$ and $h \in \mathbb{R}^n$ there exists the limit

$$(2.9) \quad \lim_{s \rightarrow -\infty} \widehat{\nu}_s(U^*(t, s)h) = \widehat{\nu}_t(h) e^{-i\langle g(t, -\infty), h \rangle + \frac{1}{2}\langle Q(t, -\infty)h, h \rangle}.$$

Being the pointwise limit of Fourier transforms of probability measures, by the Bochner Theorem the left hand side of (2.9) is the Fourier transform of a probability measure μ_t , and for each $t, t_0 \in \mathbb{R}, h \in \mathbb{R}^n$ we have

$$\widehat{\mu}_{t_0}(U^*(t, t_0)h) = \lim_{s \rightarrow -\infty} \widehat{\nu}_s(U^*(t_0, s)U^*(t, t_0)h) = \widehat{\mu}_t(h)$$

because $U^*(t_0, s)U^*(t, t_0)h = U^*(t, s)h$. Therefore, $\widehat{\mu}_t$ satisfies (2.6) with $\mu = \mu_{t_0}$. Now (2.9) implies

$$\widehat{\nu}_t(h) = e^{i\langle g(t, -\infty), h \rangle - \frac{1}{2}\langle Q(t, -\infty)h, h \rangle} \widehat{\mu}_t(h),$$

hence $\nu_t = \mathcal{N}_{g(t, -\infty), Q(t, -\infty)} \star \mu_t$. \square

The family of measures $(\mathcal{N}_{g(t, -\infty), Q(t, -\infty)})_{t \in \mathbb{R}}$ corresponds to $t_0 = 0$ and $\mu_0 = \delta_0$. For a similar result in a much more general setting (but with $f \equiv 0$) see [Dyn89, Thm. 5.1]. In the case of T -periodic coefficients, it is the unique T -periodic evolution system of measures for $P_{s,t}$, see [DPL06]. In the general case, we have the following characterization.

Lemma 2.3. *Let $(\nu_t)_{t \in \mathbb{R}}$ be an evolution system of measures for $P_{s,t}$ such that*

$$\exists \alpha > 0 : \sup_{t \in \mathbb{R}} \int_{\mathbb{R}^n} |x|^\alpha \nu_t(dx) < +\infty.$$

Then $\nu_t = \mathcal{N}_{g(t, -\infty), Q(t, -\infty)}$, for each $t \in \mathbb{R}$.

Proof. Since, by assumption, $\int_{\mathbb{R}^n} \varphi(x) \nu_t(dx) = \int_{\mathbb{R}^n} P_{s,t} \varphi(x) \nu_s(dx)$ for each $s \leq t$, it is enough to show that

$$\lim_{s \rightarrow -\infty} \int_{\mathbb{R}^n} P_{s,t} \varphi(x) \nu_s(dx) = \int_{\mathbb{R}^n} \varphi(x) \mathcal{N}_{g(t, -\infty), Q(t, -\infty)}(dx), \quad t \in \mathbb{R}, \quad \varphi \in C_b^1(\mathbb{R}^n).$$

We have

$$\begin{aligned} \int_{\mathbb{R}^n} P_{s,t} \varphi(x) \nu_s(dx) &= \int_{\mathbb{R}^n} \varphi(y) \mathcal{N}_{g(t, -\infty), Q(t, -\infty)}(dy) \\ &\quad + \int_{\mathbb{R}^n} \left(P_{s,t} \varphi(x) - \int_{\mathbb{R}^n} \varphi(y) \mathcal{N}_{g(t, -\infty), Q(t, -\infty)}(dy) \right) \nu_s(dx). \end{aligned}$$

To prove that the last integral goes to zero as $s \rightarrow -\infty$ we estimate the integrand

$$\begin{aligned} (2.10) \quad &\left| P_{s,t} \varphi(x) - \int_{\mathbb{R}^n} \varphi(y) \mathcal{N}_{g(t, -\infty), Q(t, -\infty)}(dy) \right| \\ &= \left| \int_{\mathbb{R}^n} (P_{s,t} \varphi(x) - P_{s,t} \varphi(y)) \mathcal{N}_{g(s, -\infty), Q(s, -\infty)}(dy) \right|. \end{aligned}$$

Without loss of generality we may assume that $\alpha < 1$. Fix $\omega \in (\omega_0(U), 0)$. Recalling that

$$[\psi]_{C^\alpha(\mathbb{R}^n)} \leq (2\|\psi\|_\infty)^{1-\alpha} \|D\psi\|_\infty^\alpha, \quad \psi \in C_b^1(\mathbb{R}^n),$$

and that

$$|D_x P_{s,t} \varphi(x)| = |U^*(t, s)(P_{s,t}(D\varphi))(x)| \leq M e^{\omega(t-s)} \|D\varphi\|_\infty \quad s \leq t, \quad x \in \mathbb{R}^n,$$

we get

$$|P_{s,t} \varphi(x) - P_{s,t} \varphi(y)| \leq (2\|\varphi\|_\infty)^{1-\alpha} (M e^{\omega(t-s)} \|D\varphi\|_\infty |x - y|)^\alpha := C_1 e^{\alpha\omega(t-s)} |x - y|^\alpha,$$

with C_1 independent of t, s, x, y . Therefore,

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} (P_{s,t} \varphi(x) - P_{s,t} \varphi(y)) \mathcal{N}_{g(s, -\infty), Q(s, -\infty)}(dy) \right| \\ &\leq C_1 e^{\alpha\omega(t-s)} \int_{\mathbb{R}^n} |x - y|^\alpha \mathcal{N}_{g(s, -\infty), Q(s, -\infty)}(dy). \end{aligned}$$

By the Hölder inequality we have

$$\begin{aligned} \int_{\mathbb{R}^n} |x - y|^\alpha \mathcal{N}_{g(s, -\infty), Q(s, -\infty)}(dy) &\leq \left(\int_{\mathbb{R}^n} |x - y|^2 \mathcal{N}_{g(s, -\infty), Q(s, -\infty)}(dy) \right)^{\alpha/2} \\ &= (|x - g(s, -\infty)|^2 + \text{Tr } Q(s, -\infty))^{\alpha/2} \leq C_2 (|x|^\alpha + 1), \end{aligned}$$

with C_2 independent of s, x . Replacing in (2.10), we get

$$\left| P_{s,t} \varphi(x) - \int_{\mathbb{R}^n} \varphi(y) \mathcal{N}_{g(t, -\infty), Q(t, -\infty)}(dy) \right| \leq C_1 C_2 e^{\alpha\omega(t-s)} (|x|^\alpha + 1)$$

and since $\omega < 0$ the statement follows. \square

From now on, we shall consider only the evolution system of measures

$$\nu_t := \mathcal{N}_{g(t, -\infty), Q(t, -\infty)}$$

and the corresponding invariant measure ν for \mathcal{P}_τ . Note that they satisfy

$$\sup_{t \in \mathbb{R}} \int_{\mathbb{R}^n} |x|^\alpha \nu_t(dx) < +\infty, \quad \forall \alpha > 0.$$

In contrast to the autonomous case, we cannot expect

$$\int_{\mathbb{R}^n} \mathcal{L}(s) \varphi \nu_s(dx) = 0, \quad \forall \varphi \in H^2(\mathbb{R}^n, \nu_s), s \in \mathbb{R}.$$

However, we have the following Lemma. For its proof, we recall that in the nondegenerate case $\det Q \neq 0$ the density $\rho_{m,Q}$ of $\mathcal{N}_{m,Q}$ with respect to the Lebesgue measure is given by

$$\rho_{m,Q}(y) = (2\pi)^{-\frac{n}{2}} (\det Q)^{-\frac{1}{2}} e^{-\frac{1}{2} \langle Q^{-1}(y-m), y-m \rangle}, \quad y \in \mathbb{R}^n.$$

Lemma 2.4. *Let $\varphi \in H^2(\mathbb{R}^n, \nu_s)$ for some $s \in \mathbb{R}$. Then,*

$$\int_{\mathbb{R}^n} \mathcal{L}(s) \varphi \nu_s(dx) = \int_{\mathbb{R}^n} \varphi \partial_s \rho(s, x) dx,$$

where $\rho(s, x) = \rho_{g(s, -\infty), Q(s, -\infty)}$.

Proof. Since $\text{span}\{e^{i\langle k, \cdot \rangle} : k \in \mathbb{R}^n\}$ is dense in $H^2(\mathbb{R}^n, \nu_s)$, it suffices to show that

$$\int_{\mathbb{R}^n} \mathcal{L}(s) e^{i\langle k, x \rangle} \nu_s(dx) = \int_{\mathbb{R}^n} e^{i\langle k, x \rangle} \partial_s \rho(s, x) dx, \quad k \in \mathbb{R}^n.$$

We have

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{L}(s) e^{i\langle k, x \rangle} \nu_s(dx) &= \int_{\mathbb{R}^n} \left(-\frac{1}{2} |B^* k|^2 + i \langle A(s)x, k \rangle + i \langle f(s), k \rangle \right) e^{i\langle k, x \rangle} \nu_s(dx) \\ &= -\frac{1}{2} |B^* k|^2 \widehat{\mathcal{N}}_{g(s, -\infty), Q(s, -\infty)}(k) + \langle A(s) \nabla \widehat{\mathcal{N}}_{g(s, -\infty), Q(s, -\infty)}(k), k \rangle \\ &\quad + i \langle f(s), k \rangle \widehat{\mathcal{N}}_{g(s, -\infty), Q(s, -\infty)}(k) \\ &= \left[-\frac{1}{2} |B^* k|^2 + \langle A(s) (ig(s, -\infty) - Q(s, -\infty)k), k \rangle \right. \\ &\quad \left. + i \langle f(s), k \rangle \right] \widehat{\mathcal{N}}_{g(s, -\infty), Q(s, -\infty)}(k) \\ &= \partial_s \widehat{\mathcal{N}}_{g(s, -\infty), Q(s, -\infty)}(k) = \int_{\mathbb{R}^n} e^{i\langle k, x \rangle} \partial_s \rho(s, x) dx. \end{aligned}$$

\square

We recall that $D(G_0)$ is the linear span of real and imaginary parts of the functions u of the type $u(t, x) = \Phi_j(t) e^{i\langle x, h_j(t) \rangle}$ with $\Phi_j \in C_c^1(\mathbb{R})$ and $h_j \in C_b^1(\mathbb{R}; \mathbb{R}^n)$.

Lemma 2.5. *$D(G_0)$ is dense in $L^p(\mathbb{R}^{1+n}, \nu)$, for every $p \in [1, +\infty)$.*

Proof. Since ν is a σ -finite measure on \mathbb{R}^{1+n} then the space of the continuous functions with compact support is dense in $L^p(\nu)$. Each continuous function with compact support Φ may be approximated in the sup norm (and hence in $L^p(\nu)$) by a sequence of functions that are linear combinations of products $g(t)\varphi(x)$, where both g and φ have compact support. In its turn, each continuous φ with compact support is the pointwise limit of a sequence of exponential functions φ_k such that $\|\varphi_k\|_\infty \leq \|\varphi\|_\infty$ for each k . The functions $(t, x) \mapsto g(t)\varphi_k(x)$ belong to $D(G_0)$ and approximate $g(t)\varphi(x)$ in $L^p(\nu)$. Note that if $\varphi \in L^{p_1} \cap L^{p_2}(\mathbb{R}^{1+n}, \nu)$, then the approximation is simultaneous, i.e. the same sequence approximates φ both in $L^{p_1}(\mathbb{R}^{1+n}, \nu)$ and in $L^{p_2}(\mathbb{R}^{1+n}, \nu)$. \square

Since G_0 is dissipative in $L^2(\mathbb{R}^{1+n}, \nu)$ (see the introduction) then it is closable. Let us denote the closure of G_0 by G . Then G is a dissipative, densely defined, closed operator. Moreover, it is the generator of the semigroup \mathcal{P}_τ as the next proposition shows.

Proposition 2.6. *\mathcal{P}_τ is a strongly continuous contraction semigroup in $L^2(\mathbb{R}^{1+n}, \nu)$, that leaves $D(G_0)$ invariant. Its infinitesimal generator is the closure G of G_0 . Moreover ν is an invariant measure for \mathcal{P}_τ .*

Proof. The proof is similar to the one in the T -periodic case that can be found in [DPL06]; we write it here for the reader's convenience.

Let $u \in D(G_0)$, $u(t, x) = \Phi(t)e^{i\langle x, h(t) \rangle}$ and fix $\tau > 0$. Then we have

$$\begin{aligned}
 \mathcal{P}_\tau u(t, x) &= \int_{\mathbb{R}^n} \Phi(t + \tau) e^{i\langle U(t+\tau, t)x + g(t+\tau, t) + y, h(t+\tau) \rangle} \mathcal{N}_{0, Q(t+\tau, t)}(dy) \\
 (2.11) \quad &= \Phi(t + \tau) e^{i\langle g(t+\tau, t), h(t+\tau) \rangle} e^{i\langle U(t+\tau, t)x, h(t+\tau) \rangle} e^{-\frac{1}{2} \langle Q(t+\tau, t)h(t+\tau), h(t+\tau) \rangle} \\
 &:= \Psi_\tau(t) e^{i\langle x, U^*(t+\tau, t)h(t+\tau) \rangle}.
 \end{aligned}$$

Therefore, \mathcal{P}_τ preserves $D(G_0)$, the semigroup law follows easily, as well as the strong continuity on $D(G_0)$.

Let us identify the generator of \mathcal{P}_τ as G . The domain $D(G_0)$ is contained in the domain of the infinitesimal generator L of \mathcal{P}_τ , because for $u = \phi(t)e^{i\langle x, h(t) \rangle}$ we have by (2.11)

$$\begin{aligned}
 &\frac{d}{d\tau} \mathcal{P}_\tau u|_{\tau=0} \\
 &= (\phi'(t) + i\phi(t)\langle x, h'(t) \rangle) e^{i\langle x, h(t) \rangle} - \left(\frac{1}{2} |B^*(t)h(t)|^2 + i\langle A(t)x + f(t), h(t) \rangle \right) u(t, x) \\
 &= (G_0 u)(t, x).
 \end{aligned}$$

Since $D(G_0)$ is invariant under \mathcal{P}_τ and dense in $L^2(\mathbb{R}^{1+n}, \nu)$, then it is a core for L , which means that it is dense in $D(L)$ for the graph norm. Therefore, L is the closure of G_0 . Since $L = G$ is dissipative, \mathcal{P}_τ is a contraction semigroup in $L^2(\mathbb{R}^{1+n}, \nu)$.

The fact that ν is invariant for \mathcal{P}_τ follows easily from (2.4). \square

3. SMOOTHING PROPERTIES OF THE EVOLUTION OPERATOR AND OF THE EVOLUTION SEMIGROUPS

In this section we prove estimates for the spatial derivatives of $P_{s,t}\varphi$ with $\varphi \in L^p(\mathbb{R}^n, \nu_t)$ and for the spatial derivatives of $\mathcal{P}_\tau u$, with $u \in L^p(\mathbb{R}^{1+n}, \nu)$ or $u \in L^p_\#(\mathbb{R}^{1+n}, \nu)$. In order to do so, we first obtain estimates for the spatial derivatives of the density $\rho_{U(t,s)x-g(t,s), Q(t,s)}$ of $\mathcal{N}_{U(t,s)x-g(t,s), Q(t,s)}$. For notational reasons we suppress that ρ depends on t, s and x and we shortly write $\rho(y)$ for $\rho_{U(t,s)x-g(t,s), Q(t,s)}(y)$.

Lemma 3.1. *Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$ and let $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$, or $(p, q) = (1, +\infty)$. Then there exists $C > 0$ such that*

$$(3.1) \quad \|\rho^{-\frac{1}{p}} D_x^\alpha \rho\|_{L^q(\mathbb{R}^n, dx)} \leq C \|Q^{-\frac{1}{2}}(t, s)\|^k \|U(t, s)\|^k, \quad x \in \mathbb{R}^n, \quad t > s.$$

Proof. Since $D_x \rho = \rho \cdot U^*(t, s) Q^{-1}(t, s)(y - U(t, s)x + g(t, s))$, differentiating further we obtain that $|D_x^\alpha \rho| \leq \rho \cdot P_\alpha(\|\mathcal{A}_1(t, s, x, y)\|, \|\mathcal{A}_2(t, s, x, y)\|)$, where

$$\begin{aligned} \mathcal{A}_1(t, s, x, y) &= U(t, s)^* Q^{-1}(t, s)(y - U(t, s)x + g(t, s)), & s, t \in \mathbb{R}, \quad x, y \in \mathbb{R}^n, \\ \mathcal{A}_2(t, s, x, y) &= -U^*(t, s) Q^{-1}(t, s) U(t, s), & s, t \in \mathbb{R}, \quad x, y \in \mathbb{R}^n. \end{aligned}$$

and $P_\alpha(\xi, \eta) = \sum_{i+2j=k} \beta_{ij} \xi^i \eta^j$ for some $\beta_{ij} \in \mathbb{R}$.

The statement follows now immediately if $p = 1, q = \infty$. If $p > 1$ then

$$\int_{\mathbb{R}^n} |\rho^{-\frac{1}{p}}(y) D_x^\alpha \rho(y)|^q dy \leq C \sum_{i,j \in \mathbb{N}_0, i+2j=k} \int_{\mathbb{R}^n} \rho(y) \|\mathcal{A}_1(t, s, x, y)\|^{iq} \|\mathcal{A}_2(t, s, x, y)\|^{jq} dy.$$

By the substitution $y = Q(t, s)^{\frac{1}{2}} \eta + U(t, s)x + g(t, s)$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \rho(y) \|\mathcal{A}_1(t, s, x, y)\|^{iq} \|\mathcal{A}_2(t, s, x, y)\|^{jq} dy \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp(-|\eta|^2/2) \|Q^{-\frac{1}{2}}(t, s) U(t, s)\|^{iq} |\eta|^{iq} \|U^*(t, s) Q^{-1}(t, s) U(t, s)\|^{jq} d\eta \\ &\leq C \|Q^{-\frac{1}{2}}(t, s) U(t, s)\|^{iq} \|U^*(t, s) Q^{-1}(t, s) U(t, s)\|^{jq} \leq C \|Q^{-\frac{1}{2}}(t, s)\|^{iq+2jq} \|U(t, s)\|^{iq+2jq} \\ &= C \|Q^{-\frac{1}{2}}(t, s)\|^{kq} \|U(t, s)\|^{kq}. \end{aligned}$$

Summing up, the proof is complete. \square

The next lemma provides estimates for $Q^{-\frac{1}{2}}(t, s)$.

Lemma 3.2. *There exist $C, \delta > 0$ such that*

$$\|Q^{-\frac{1}{2}}(t, s)\| \leq \begin{cases} C(t-s)^{-\frac{1}{2}}, & 0 < t-s < \delta, \\ C, & t-s \geq \delta. \end{cases}$$

Proof. Let $x \in \mathbb{R}^n$. Then, by (1.8),

$$\begin{aligned} \langle Q(t, s)x, x \rangle &= \int_s^t \langle U(t, r)B(r)B^*(r)U^*(t, r)x, x \rangle \, dr = \int_s^t \|B^*(r)U^*(t, r)x\|^2 \, dr \\ &\geq \mu_0 \int_s^t \|U^*(t, r)x\|^2 \, dr. \end{aligned}$$

Since $\|U^*(t, r)x - x\| \leq \frac{1}{2}\|x\|$ for $t - r < \delta$ with some $\delta > 0$, independent of t, r , and x , we obtain

$$\langle Q(t, s)x, x \rangle \geq \frac{\mu_0}{4}(t - s)\|x\|^2, \quad 0 < t - s < \delta.$$

Similarly, for $t - s \geq \delta$, we have

$$\langle Q(t, s)x, x \rangle = \int_s^t \|B^*(r)U^*(t, r)x\|^2 \, dr \geq \int_{t-\delta}^t \|B^*(r)U^*(t, r)x\|^2 \, dr \geq \frac{\mu_0}{4}\delta\|x\|^2.$$

□

By (1.9) we obtain, for $\omega \in (\omega_0(U), 0)$,

$$\begin{aligned} \langle Q(t, s)x, x \rangle &= \int_s^t \|B^*(r)U^*(t, r)x\|^2 \, dr \leq CM \int_s^t e^{2\omega(t-r)}\|x\|^2 \, dr \\ (3.2) \quad &\leq CM \frac{1 - e^{\omega(t-s)}}{2|\omega|} \|x\|^2, \quad t > s. \end{aligned}$$

Hence, $\|Q^{-\frac{1}{2}}(t, s)\|$ does not decay for $t - s \rightarrow \infty$. In other words, the estimate of Lemma 3.2 is optimal for $t - s \rightarrow \infty$.

Now we are in the position to prove estimates for the spatial derivatives of $P_{s,t}\varphi$, for each $\varphi \in L^p(\mathbb{R}^n, \nu_t)$.

Lemma 3.3. *Let $\alpha \in \mathbb{N}_0^n$ and $p \in [1, \infty)$. For $\omega > \omega_0(U)$ there exist $C, \delta > 0$ such that,*

$$\|D_x^\alpha P_{s,t}\|_{\mathcal{L}(L^p(\mathbb{R}^n, \nu_t), L^p(\mathbb{R}^n, \nu_s))} \leq \begin{cases} C(t-s)^{|\alpha|/2} e^{\omega|\alpha|(t-s)}, & 0 < t-s < \delta, \\ C e^{\omega|\alpha|(t-s)}, & t-s > \delta. \end{cases}$$

Proof. Since $C_b(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, \nu_t)$ for any $t \in \mathbb{R}$, it is enough to estimate $\|D_x^\alpha P_{s,t}\|_{L^p(\mathbb{R}^n, \nu_s)}$ for $\varphi \in C_b(\mathbb{R}^n)$. Hölder's inequality and Lemma 3.1 yield

$$\begin{aligned} |D_x^\alpha P_{s,t}\varphi(x)|^p &= \left| D_x^\alpha \int_{\mathbb{R}^n} \varphi(y + g(t, s))\rho(y) \, dy \right|^p \\ &\leq \int_{\mathbb{R}^n} |\varphi(y + g(t, s))|^p \rho(y) \, dy \|\rho^{-\frac{1}{p}} D_x^\alpha \rho\|_{L^q(\mathbb{R}^n, dx)}^p \\ &= (P_{s,t}|\varphi|^p)(x) \|\rho^{-\frac{1}{p}} D_x^\alpha \rho\|_{L^q(\mathbb{R}^n, dx)}^p \\ &\leq C (P_{s,t}|\varphi|^p)(x) \|Q^{-\frac{1}{2}}(t, s)\|^k \|U(t, s)\|^k, \end{aligned}$$

where $1/p + 1/q = 1$. Hence, it follows from (2.4) that

$$\begin{aligned} \|D_x^\alpha P_{s,t} \varphi\|_{L^p(\nu_t)}^p &\leq C \|Q^{-\frac{1}{2}}(t,s)\|^k \|U(t,s)\|^k \int_{\mathbb{R}^n} P_{s,t} |\varphi|^p(x) \nu_t(dx) \\ &= C \|Q^{-\frac{1}{2}}(t,s)\|^k \|U(t,s)\|^k \int_{\mathbb{R}^n} |\varphi|^p(x) \nu_s(dx) \\ &= C \|Q^{-\frac{1}{2}}(t,s)\|^k \|U(t,s)\|^k \|\varphi\|_{L^p(\mathbb{R}^n, \nu_s)}^p. \end{aligned}$$

Here, we have used that $|\varphi|^p \in C_b(\mathbb{R}^n)$. Now, Lemma 3.2 and (1.9) yield the assertion. \square

Thanks to the representation (1.14), the smoothing properties of the evolution operator $P_{s,t}$, given in Lemma 3.3, yield smoothing properties of the semigroups $(\mathcal{P}_\tau)_{\tau \geq 0}$ and $(\mathcal{P}_\tau^\#)_{\tau \geq 0}$.

Lemma 3.4. *Let $\alpha \in \mathbb{N}_0^n$ and $p \in [1, \infty)$. For $\omega > \omega_0(U)$ there exist $C, \delta > 0$, such that*

$$\|D_x^\alpha \mathcal{P}_\tau\|_{\mathcal{L}(L^p(\mathbb{R}^{1+n}, \nu))} \leq \begin{cases} C \tau^{-\frac{|\alpha|}{2}} e^{\omega|\alpha|\tau} & , \quad 0 < t - s < \delta, \\ C e^{\omega|\alpha|\tau} & , \quad t - s \geq \delta. \end{cases}$$

Proof. By Lemma 3.3, there exist $C, \delta > 0$, such that

$$\begin{aligned} \|D_x^\alpha \mathcal{P}_\tau u\|_{L^p(\mathbb{R}^{1+n}, \nu)}^p &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} |D_x^\alpha \mathcal{P}_\tau u(t, x)|^p \nu_t(dx) dt = \int_{\mathbb{R}} \int_{\mathbb{R}^n} |D_x^\alpha P_{t, t+\tau} u(t + \tau, x)|^p \nu_t(dx) dt \\ &\leq C K(\tau)^p \int_{\mathbb{R}} \int_{\mathbb{R}^n} |u(t + \tau, x)|^p \nu_{t+\tau}(dx) dt, \quad u \in L^p(\mathbb{R}^{1+n}, \nu), \end{aligned}$$

where

$$K(\tau) := \begin{cases} \tau^{-\frac{|\alpha|}{2}} e^{\omega|\alpha|\tau} & , \quad 0 < t - s < \delta, \\ e^{\omega|\alpha|\tau} & , \quad t - s \geq \delta. \end{cases}$$

Now, the lemma follows from the substitution $s = t + \tau$. Indeed, for each $v \in L^p(\mathbb{R}^{1+n}, \nu)$ we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |v(t + \tau, x)|^p \nu_{t+\tau}(dx) dt = \int_{\mathbb{R}} \int_{\mathbb{R}^n} |v(s, x)|^p \nu_s(dx) ds = \|v\|_{L^p(\mathbb{R}^{1+n}, \nu)}^p.$$

\square

The proof of Lemma 3.4 can be easily carried over to the T -periodic case and it is therefore omitted.

Lemma 3.5. *Let $\alpha \in \mathbb{N}_0^n$ and $p \in [1, \infty)$. For each $\omega > \omega_0(U)$ there exist $C, \delta > 0$, such that*

$$\|D_x^\alpha \mathcal{P}_\tau\|_{\mathcal{L}(L_\#^p(\mathbb{R}^{1+n}, \nu))} \leq \begin{cases} C \tau^{-\frac{|\alpha|}{2}} e^{\omega|\alpha|\tau} & , \quad 0 < t - s < \delta, \\ C e^{\omega|\alpha|\tau} & , \quad t - s \geq \delta. \end{cases}$$

4. THE SPACES $H^{k,s}(\mathbb{R}^{1+n}, \nu)$ AND $H_{\#}^{k,s}(\mathbb{R}^{1+n}, \nu)$

In this section we show some properties of the spaces $H^{k,s}(\mathbb{R}^{1+n}, \nu)$ and $H_{\#}^{k,s}(\mathbb{R}^{1+n}, \nu)$ defined in the introduction.

We recall that for any Gaussian measure $\mathcal{N}_{m,Q}$ and for any $k \in \mathbb{N}_0$, the space $H^k(\mathbb{R}^n, \mathcal{N}_{m,Q})$ is defined as

$$H^k(\mathbb{R}^n, \mathcal{N}_{m,Q}) := \{f \in L^2(\mathbb{R}^n, \mathcal{N}_{m,Q}) : \exists D_{\beta} f \in L^2(\mathbb{R}^n, \mathcal{N}_{m,Q}), |\beta| \leq k\}.$$

Then, $H^k(\mathbb{R}^n, \mathcal{N}_{m,Q})$ equipped with its natural norm is a Hilbert space.

We first show that $D(G_0)$ is dense in $H^{1,2}(\mathbb{R}^{1+n}, \nu)$.

Lemma 4.1. *$D(G_0)$ is dense in $H^{1,2}(\mathbb{R}^{1+n}, \nu)$.*

Proof. Let $v \in C_c^{\infty}(\mathbb{R}^{1+n})$. We choose $R_0 > 0$ and $S > 0$ such that $\text{supp } v \subset \Omega_{S,R_0}$, where $\Omega_{S,R_0} := (-S, S) \times (-R_0, R_0)^n$. For $R > R_0$ and $k \in \mathbb{Z}^n$ we set

$$v_{R,l}(t, x) = \sum_{|k|=0}^l a_{R,k}(t) e^{i \frac{R}{\pi} \langle k, x \rangle},$$

where $a_{R,k}(t) := \frac{1}{(2R)^n} \int_{(-R,R)^n} v(t, x) e^{-i \frac{R}{\pi} \langle k, x \rangle} dx$.

We will show that for each $\varepsilon > 0$ there exists $R > R_0$ and $l \in \mathbb{N}$ such that

$$\|v - v_{R,l}\|_{H^{1,2}(\mathbb{R}^{1+n}, \nu)} \leq \varepsilon.$$

Since v is smooth and compactly supported, there exists $K > 0$ such that for each $R \geq R_0$ and $k \in \mathbb{Z}^n$ we have

$$\|\partial_t v_{R,l}\|_{L^{\infty}(\mathbb{R}^{1+n})}^2 + \sum_{|\alpha|=0}^2 \|D_x^{\alpha} v_{R,l}\|_{L^{\infty}(\mathbb{R}^{1+n})}^2 \leq K.$$

Let us fix $R \geq R_0$ such that $K\nu((-S, S) \times \mathbb{R}^n \setminus \Omega_{S,R}) \leq \varepsilon/2$. As $l \rightarrow \infty$, the sequences $(v_{R,l})$, $(\partial_t v_{R,l})$, $(D_x^{\alpha} v_{R,l})$ converge uniformly on $\Omega_{S,R}$ to v , $\partial_t v$, $D_x^{\alpha} v$, respectively. Therefore, there exists $l \in \mathbb{N}$ such that

$$\sum_{|\alpha|=0}^2 \|D_x^{\alpha} v_{R,l} - D_x^{\alpha} v\|_{L^{\infty}(\Omega_{S,R})}^2 + \|\partial_t v_{R,l} - \partial_t v\|_{L^{\infty}(\Omega_{S,R})}^2 \leq \frac{\varepsilon}{2\nu(\Omega_{S,R})}.$$

For such l we have

$$\|v - v_{R,l}\|_{H^{1,2}(\mathbb{R}^{1+n}, \nu)}^2 = \|v - v_{R,l}\|_{H^{1,2}(\Omega_{S,R}, \nu)}^2 + \|v_{R,l}\|_{H^{1,2}(\Omega_{S,R}^c, \nu)}^2 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $C_c^{\infty}(\mathbb{R}^{1+n})$ is dense in $H^{1,2}(\mathbb{R}^{1+n}, \nu)$, the proof is complete. \square

The corresponding result for T -periodic spaces reads as follows.

Lemma 4.2. *$D(G_0^{\#})$ is dense in $L_{\#}^2(\mathbb{R}^{1+n}, \nu)$ and in $H_{\#}^{1,2}(\mathbb{R}^{1+n}, \nu)$.*

Proof. Let

$$v \in C_{c,\#}^{\infty} := \{u \in C^{\infty}(\mathbb{R}^{1+n}) : u(t, x) = u(T + t, x) \text{ for all } t \in \mathbb{R}, x \in \mathbb{R}^n, \text{ and } \text{supp } u(t, \cdot) \subset (-R_0, R_0)^n \text{ for all } t \in \mathbb{R} \text{ and some } R_0 > 0\}.$$

We define

$$K := \|\partial_t v\|_{L^\infty(\mathbb{R}^{1+n})} + \sum_{|\alpha|=0}^2 \|D_x^\alpha v\|_{L^\infty(\mathbb{R}^{1+n})}.$$

and, for $R > R_0$, we set

$$v_{R,l}(t, x) = \sum_{|k|=-l}^l a_{R,k}(t) e^{i \frac{R}{\pi} \langle k, x \rangle},$$

where $a_{R,k}(t) := \frac{1}{(2R)^n} \int_{-R}^R v(t, x) e^{i \frac{R}{\pi} \langle k, x \rangle} dx$. Clearly, $a_{R,k}$ is T -periodic. As in the proof of Lemma 4.1, it follows that for $\varepsilon \in (0, K)$ there exists $R > R_0$ and $l \in \mathbb{N}$ such that

$$\|v - v_{R,l}\|_{H_{\#}^{1,2}(\mathbb{R}^{1+n}, \nu)} \leq \varepsilon.$$

Since $C_{c,\#}^\infty((0, T) \times \mathbb{R}^n)$ is dense in $H_{\#}^{1,2}((0, T) \times \mathbb{R}^n, \nu)$, the proof is complete. \square

Let $t_0 \in \mathbb{R}$. In the following we denote the product measure of the one dimensional Lebesgue measure and ν_{t_0} by $dt \times \nu_{t_0}$.

Lemma 4.3. (a) *There exists an isomorphism*

$$\mathcal{T} : L^2(\mathbb{R}^{1+n}, \nu) \rightarrow L^2(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1}),$$

such that, for $k = 0$, $s \in \mathbb{N}_0$ and for $k = 1$, $s = 2$, $\mathcal{T}|_{H^{k,s}(\mathbb{R}^n, \nu)}$ is an isomorphism from $H^{k,s}(\mathbb{R}^{1+n}, \nu)$ onto $H^{k,s}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})$.

(b) *Let $t_0 \in \mathbb{R}$. Then, there exists an isomorphism*

$$\mathcal{T}_{t_0} : L^2(\mathbb{R}^{1+n}, \nu) \rightarrow L^2(\mathbb{R}^{1+n}, dt \times \nu_{t_0}),$$

such that, for $k = 0$, $s \in \mathbb{N}_0$ and for $k = 1$, $s = 2$, $\mathcal{T}_{t_0}|_{H^{k,s}(\mathbb{R}^n, \nu)}$ is an isomorphism from $H^{k,s}(\mathbb{R}^{1+n}, \nu)$ onto $H^{k,s}(\mathbb{R}^{1+n}, dt \times \nu_{t_0})$.

Proof. For $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ define

$$(\mathcal{T}u)(t, x) := u\left(t, Q^{\frac{1}{2}}(t, -\infty)x + g(t, -\infty)\right).$$

By substitution, we obtain

$$\|u\|_{L^2(\mathbb{R}^{1+n}, \nu)} = \|\mathcal{T}u\|_{L^2(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})}, \quad u \in L^2(\mathbb{R}^{1+n}, \nu).$$

Moreover, for $l \in \mathbb{N}$ and for arbitrary integers $\alpha_1, \dots, \alpha_l \in \{1, \dots, n\}$ we have

$$\frac{\partial \mathcal{T}u}{\partial x_{\alpha_1} \dots \partial x_{\alpha_l}}(x) = \sum_{\beta_i \in \{1, \dots, n\}} \frac{\partial u}{\partial \beta_1 \dots \partial \beta_l}(Q^{\frac{1}{2}}(t, -\infty)x + g(t, -\infty)) \prod_{i=1}^l (Q^{\frac{1}{2}}(t, -\infty)_{\beta_i \alpha_i})$$

so that, for $u \in H^{0,k}(\mathbb{R}^{1+n}, \nu)$,

$$\|D_x^k \mathcal{T}u\|_{L^2(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})} \leq \|Q^{\frac{1}{2}}(t, -\infty)\|^k \|D_x^k u\|_{L^2(\mathbb{R}^{1+n}, \nu)}.$$

and similarly, for $u \in H^{0,k}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})$,

$$\|D_x^k \mathcal{T}^{-1}u\|_{L^2(\mathbb{R}^{1+n}, \nu)} \leq \|Q^{-\frac{1}{2}}(t, -\infty)\|^k \|D_x^k u\|_{L^2(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})}.$$

Since the norms $\|Q^{\frac{1}{2}}(t, -\infty)\|$ and $\|Q^{-\frac{1}{2}}(t, -\infty)\|$ are bounded by a constant independent of t , then \mathcal{T} is an isomorphism from $H^{0,k}(\mathbb{R}^{1+n}, \nu)$ to $H^{0,k}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})$ for $k \in \mathbb{N}_0$.

For $\varphi \in H^{1,2}(\mathbb{R}^{1+n}, \nu)$ we have

$$\begin{aligned} \partial_t(\mathcal{T}\varphi)(t, x) &= \partial_t\varphi(t, Q^{\frac{1}{2}}(t, -\infty)x - g(t, -\infty)) = (\partial_t\varphi)(t, Q^{\frac{1}{2}}(t, -\infty)x - g(t, -\infty)) \\ &\quad + \langle (\nabla_x \varphi)(t, Q^{\frac{1}{2}}(t, -\infty)x - g(t, -\infty)), \partial_t Q^{\frac{1}{2}}(t, -\infty)x - \partial_t g(t, -\infty) \rangle. \end{aligned}$$

Clearly, $\partial_t g(t, -\infty) = f(t) + \int_{-\infty}^t A(t)U(t, r)f(r) \, dr$ is uniformly bounded for $t \in \mathbb{R}$. Moreover, the representation

$$Q^{\frac{1}{2}}(t, -\infty) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{\frac{1}{2}} R(\lambda, Q(t, -\infty)) \, d\lambda$$

for a suitable path Γ yields

$$\|\partial_t Q^{\frac{1}{2}}(t, -\infty)\| \leq C, \quad t \in \mathbb{R},$$

thanks to the uniform boundedness of $\partial_t R(\lambda, Q(t, -\infty))$ for $\lambda \in \Gamma$ and $t \in \mathbb{R}$. The latter follows from the boundedness of $\|A(t)\|$, $\|B(t)\|$, and $\|Q(t, -\infty)\|$, see (3.2). Moreover, an easy computation (see e.g. [Lun97b, Lemma 2.1], or [MPRS02, Lemma 2.3]) shows that there exists $C > 0$ such that for any $M \in \mathcal{L}(\mathbb{R}^n)$ and $t \in \mathbb{R}$, $\psi \in H^2(\mathbb{R}^n, \nu_t)$

$$(4.1) \quad \|\langle M \cdot, D_x \psi \rangle\|_{L^2(\mathbb{R}^n, \nu_t)} \leq C \|M\| \cdot \|\psi\|_{H^2(\mathbb{R}^n, \nu_t)}.$$

Therefore, it follows that

$$\|\mathcal{T}\varphi\|_{H^{1,2}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})} \leq C \|\varphi\|_{H^{1,2}(\mathbb{R}^{1+n}, \nu)}, \quad \varphi \in H^{1,2}(\mathbb{R}^{1+n}, \nu).$$

Similarly, we obtain

$$\|\mathcal{T}^{-1}\varphi\|_{H^{1,2}(\mathbb{R}^{1+n}, \nu)} \leq C \|\varphi\|_{H^{1,2}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})}, \quad \varphi \in H^{1,2}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1}).$$

This proves (a).

Setting

$$(\mathcal{T}_{t_0} u)(t, x) := u\left(t, Q^{\frac{1}{2}}(t, -\infty)Q^{-\frac{1}{2}}(t_0, -\infty)(x - g(t_0, -\infty)) + g(t, -\infty)\right),$$

assertion (b) follows as above. \square

There is a corresponding result for T-periodic spaces as well. Since the proof of the following lemma is similar to the proof of Lemma 4.3 it is omitted.

Lemma 4.4. (a) *There exists an isomorphism*

$$\mathcal{T}_{\#} : L_{\#}^2(\mathbb{R}^{1+n}, \nu) \rightarrow L_{\#}^2(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1}),$$

such that, for $k = 0$ and $s \in \mathbb{N}_0$ and $k = 1$ and $s = 2$, $\mathcal{T}_{\#}|_{H_{\#}^{k,s}(\mathbb{R}^{1+n}, \nu)}$ is an isomorphism from $H_{\#}^{k,s}(\mathbb{R}^{1+n}, \nu)$ onto $H_{\#}^{k,s}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})$.

(b) *Let $t_0 \in \mathbb{R}$. Then, there exists an isomorphism*

$$\mathcal{T}_{\#, t_0} : L_{\#}^2(\mathbb{R}^{1+n}, \nu) \rightarrow L_{\#}^2(\mathbb{R}^{1+n}, dt \times \nu_{t_0}),$$

such that, for $k = 0$ and $s \in \mathbb{N}_0$ and $k = 1$ and $s = 2$, $\mathcal{T}_{\#, t_0}|_{H_{\#}^{k,s}(\mathbb{R}^{1+n}, \nu)}$ is an isomorphism from $H_{\#}^{k,s}(\mathbb{R}^{1+n}, \nu)$ onto $H_{\#}^{k,s}(\mathbb{R}^{1+n}, dt \times \nu_{t_0})$.

Next, we give a characterization of some real interpolation spaces between $L^2(\mathbb{R}^{1+n}, \nu)$ and $H^{0,s}(\mathbb{R}^{1+n}, \nu)$.

Proposition 4.5. *Let $r, s \in \mathbb{N}$ with $0 < r < s$. Then,*

$$(L^2(\mathbb{R}^{1+n}, \nu), H^{0,s}(\mathbb{R}^{1+n}, \nu))_{\frac{r}{s}, 2} = H^{0,r}(\mathbb{R}^{1+n}, \nu).$$

Proof. Let \mathcal{T} be as in Lemma 4.3. Then,

$$\begin{aligned} & (L^2(\mathbb{R}^{1+n}, \nu), H^{0,s}(\mathbb{R}^{1+n}, \nu))_{\frac{r}{s}, 2} \\ &= (\mathcal{T}^{-1}L^2(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1}), \mathcal{T}^{-1}H^{0,s}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1}))_{\frac{r}{s}, 2} \\ &= \mathcal{T}^{-1}(L^2(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1}), H^{0,s}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1}))_{\frac{r}{s}, 2}. \end{aligned}$$

On the other hand, by [Tri78, Theorem 1.18.4],

$$\begin{aligned} & (L^2(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1}), H^{0,s}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1}))_{\frac{r}{s}, 2} \\ &= (L^2(\mathbb{R}, L^2(\mathbb{R}^n, \mathcal{N}_{0,1})), L^2(\mathbb{R}, H^s(\mathbb{R}^n, \mathcal{N}_{0,1})))_{\frac{r}{s}, 2} \\ &= L^2\left(\mathbb{R}, (L^2(\mathbb{R}^n, \mathcal{N}_{0,1}), H^s(\mathbb{R}^n, \mathcal{N}_{0,1}))_{\frac{r}{s}, 2}\right). \end{aligned}$$

The real interpolation spaces between L^2 and H^s spaces with respect to the Gaussian measure $\mathcal{N}_{0,1}$ are known, see e.g. [FL06, Proposition 4]. More precisely, we have

$$H^l(\mathbb{R}^n, \mathcal{N}_{0,1}) = \left(L^2(\mathbb{R}^n, \mathcal{N}_{0,1}), H^k(\mathbb{R}^n, \mathcal{N}_{0,1})\right)_{l/k, 2},$$

for each $k, l \in \mathbb{N}$ with $0 < l < k$, and, therefore, we get

$$\begin{aligned} (L^2(\mathbb{R}^{1+n}, \nu), H^{0,s}(\mathbb{R}^{1+n}, \nu))_{\frac{r}{s}, 2} &= \mathcal{T}^{-1}L^2(\mathbb{R}, H^r(\mathbb{R}^n, \mathcal{N}_{0,1})) = \mathcal{T}^{-1}H^{0,r}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1}) \\ &= H^{0,r}(\mathbb{R}^{1+n}, \nu). \end{aligned}$$

□

Again, there is a corresponding result for T -periodic spaces. Since the proof of Proposition 4.5 carries over to the T -periodic case with only minor modifications, the proof of the next lemma is omitted.

Proposition 4.6. *Let $r, s \in \mathbb{N}$ with $0 < r < s$. Then,*

$$(L^2_{\#}(\mathbb{R}^{1+n}, \nu), H^{0,s}_{\#}(\mathbb{R}^{1+n}, \nu))_{\frac{r}{s}, 2} = H^{0,r}_{\#}(\mathbb{R}^{1+n}, \nu).$$

5. THE DOMAINS OF G AND OF $G_{\#}$

The proofs of Theorem 1.1 and Theorem 1.2 are based on the following abstract interpolation result (see [Lun99, Theorem 2.5]).

Proposition 5.1. *Let $T(t)$ be a semigroup on some Banach space X with generator $L : D(L) \rightarrow X$. Assume that there exists a Banach space $E \subset X$ and $m \in \mathbb{N}$, $0 < \beta < 1$, $\omega \in \mathbb{R}$, $C > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(X, E)} \leq Ce^{\omega t} t^{-m\beta}, \quad t > 0,$$

and for every $x \in X$, $t \mapsto T(t)x$ is measurable with values in E . Then $E \in J_{\beta}(X, D(L^m))$, so that $(X, D(L^m))_{\theta, p} \subset (X, E)_{\theta/\beta, p}$, for every $\theta \in (0, \beta)$, $p \in [1, \infty]$.

Indeed, we apply this proposition taking $X = L^2(\mathbb{R}^{1+n}, \nu)$, $T(t) = \mathcal{P}_t$ and $E = H^{0,k}(\mathbb{R}^{1+n}, \nu)$, or $X = L^2_{\#}(\mathbb{R}^{1+n}, \nu)$, $T(t) = \mathcal{P}_t$ and $E = H^{0,k}_{\#}(\mathbb{R}^{1+n}, \nu)$ in the periodic case. With these choices we have optimal blow-up estimates for the norms $\|T(t)\|_{\mathcal{L}(X,E)}$ as $t \rightarrow 0$, given by Lemmas 3.4 and 3.5. Since the real interpolation spaces between X and E have been characterized in Propositions 4.5 and 4.6, the main step of the proof of Theorem 1.1 and Theorem 1.2 follows.

5.1. Proof of Theorem 1.1. We first prove the continuous embedding

$$(5.1) \quad D(G) \subset H^{1,2}(\mathbb{R}^{1+n}, \nu).$$

We use Proposition 5.1 with $X = L^2(\mathbb{R}^{1+n}, \nu)$ and $E = H^{0,4}(\mathbb{R}^{1+n}, \nu)$. From Lemma 3.4 it follows that there are C, ω , such that

$$\|\mathcal{P}_\tau\|_{\mathcal{L}(X,E)} \leq C e^{\omega\tau} \tau^{-2}, \quad \tau > 0.$$

Choosing $m = 4$, $\theta = \frac{1}{4}$ and $\beta = \frac{1}{2}$, Proposition 5.1 yields

$$(L^2(\mathbb{R}^{1+n}, \nu), D(G - I)^4)_{\frac{1}{4},2} \subset (L^2(\mathbb{R}^{1+n}, \nu), H^{0,4}(\mathbb{R}^{1+n}, \nu))_{\frac{1}{2},2}.$$

Since $L^2(\mathbb{R}^{1+n}, \nu)$ is a Hilbert space and $G - I$ is dissipative and invertible, [Kat62, Theorem 5] and [Tri78, Theorem 1.15.3] yield $(L^2(\mathbb{R}^{1+n}, \nu), D(G - I)^4)_{\frac{1}{4},2} = D(G - I) = D(G)$. Therefore, Proposition 4.5 implies

$$(5.2) \quad D(G) \subset H^{0,2}(\mathbb{R}^{1+n}, \nu).$$

Now, let $u \in D(G_0)$ and set

$$\psi(t, x) := (Gu)(t, x) = \frac{\partial}{\partial t} u(t, x) + \mathcal{L}(t)u(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

It follows from (4.1), that there exists $C_1 > 0$, independent of u , such that

$$\|\mathcal{L}(t)u(t)\|_{L^2(\mathbb{R}^n, \nu_t)} \leq C_1 \|u(t, \cdot)\|_{H^2(\mathbb{R}^n, \nu_t)}, \quad t \in \mathbb{R}.$$

Moreover by (5.2), $\|\mathcal{L}(\cdot)u\|_{L^2(\mathbb{R}^{1+n}, \nu)} \leq C_2 (\|u\|_{L^2(\mathbb{R}^{1+n}, \nu)} + \|Gu\|_{L^2(\mathbb{R}^{1+n}, \nu)})$, where $C_2 > 0$ is independent of u . Writing $\frac{\partial}{\partial t} u = \psi - \mathcal{L}(\cdot)u$, we obtain

$$\begin{aligned} \|u_t\|_{L^2(\mathbb{R}^n, \nu)} &\leq \|\psi\|_{L^2(\mathbb{R}^{1+n}, \nu)} + \|\mathcal{L}(\cdot)u\|_{L^2(\mathbb{R}^{1+n}, \nu)} \\ &\leq \|Gu\|_{L^2(\mathbb{R}^{1+n}, \nu)} + C_1 C_2 (\|u\|_{L^2(\mathbb{R}^n, \nu)} + \|Gu\|_{L^2(\mathbb{R}^{1+n}, \nu)}) = 2C \|Gu\|_{L^2(\mathbb{R}^{1+n}, \nu)}. \end{aligned}$$

Putting together this estimate and (5.2) we get

$$\|u\|_{H^{1,2}(\mathbb{R}^{1+n}, \nu)} \leq C_3 (\|u\|_{L^2(\mathbb{R}^n, \nu)} + \|Gu\|_{L^2(\mathbb{R}^{1+n}, \nu)})$$

with C_3 independent of u . Since $D(G_0)$ is a core of $D(G)$, the proof of (5.1) is complete. Moreover, since, by Lemma 4.1, $D(G_0)$ is dense in $H^{1,2}(\mathbb{R}^{1+n}, \nu)$, we have $D(G) = H^{1,2}(\mathbb{R}^{1+n}, \nu)$.

Now let us prove the second equality. The inclusion “ \subset ” is obvious. Let

$$u \in \left\{ u \in H^{1,2}_{\text{loc}}(\mathbb{R}^{1+n}) \cap L^2(\mathbb{R}^{1+n}, \nu) : \mathcal{G}u \in L^2(\mathbb{R}^{1+n}, \nu) \right\},$$

fix $\lambda > 0$ and set $\psi := \lambda u - \mathcal{G}u$. Then $v := u - R(\lambda, G)\psi$ satisfies $\lambda v - \mathcal{G}v = 0$. We will prove that $v \equiv 0$, and hence $u \in D(G)$, provided λ is large enough.

In order to do so, let $\varphi \in C_c^\infty(\mathbb{R}^{1+n})$ be such that $\varphi(\cdot, \cdot) \equiv 1$ on $[-1, 1] \times B(0, 1)$ and $\varphi(\cdot, \cdot) \equiv 0$ outside $[-2, 2] \times B(0, 2)$. Then $\varphi_k(t, x) := \varphi(t/k, x/k)$ satisfies $\varphi_k(t, x) \rightarrow 1$ for $k \rightarrow \infty$ and $|\mathbf{D}_x \varphi_k(t, x)| \leq \|\mathbf{D}_x \varphi\|_\infty$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

A direct calculation yields

$$(5.3) \quad \mathcal{G}(gh) = g\mathcal{G}h + h\mathcal{G}g + \langle B^* \mathbf{D}_x g, B^* \mathbf{D}_x h \rangle, \quad g, h \in H_{\text{loc}}^{1,2}(\mathbb{R}^{1+n}).$$

Hence, we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{1+n}} (\lambda v - \mathcal{G}v) \varphi_k^2 v \, d\nu = \lambda \|\varphi_k v\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 - \int_{\mathbb{R}^{1+n}} \varphi_k \mathcal{G}v \varphi_k v \, d\nu \\ &= \lambda \|\varphi_k v\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 - \int_{\mathbb{R}^{1+n}} \mathcal{G}(\varphi_k v) \varphi_k v \, d\nu + \int_{\mathbb{R}^{1+n}} (\mathcal{G}\varphi_k) v \varphi_k v \, d\nu \\ &\quad + \int_{\mathbb{R}^{1+n}} \langle B^* \mathbf{D}_x \varphi_k, B^* \mathbf{D}_x v \rangle \varphi_k v \, d\nu. \end{aligned}$$

Since $\int_{\mathbb{R}^{1+n}} \mathcal{G}g \, d\nu = 0$ for each $g \in D(G_0)$, it follows from (5.3) that

$$(5.4) \quad \int_{\mathbb{R}^{1+n}} g \mathcal{G}h \, d\nu + \int_{\mathbb{R}^{1+n}} h \mathcal{G}g \, d\nu + \int_{\mathbb{R}^{1+n}} \langle B^* \mathbf{D}_x g, B^* \mathbf{D}_x h \rangle \, d\nu = \int_{\mathbb{R}^{1+n}} G(gh) \, d\nu = 0$$

for $g, h \in D(G_0)$. Note that (5.4) also holds for $g, h \in D(G)$, since $D(G_0)$ is a core of $D(G)$. In particular, since $\varphi_k v \in D(G)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^{1+n}} \mathcal{G}(\varphi_k v) \varphi_k v \, d\nu &= -\frac{1}{2} \int_{\mathbb{R}^{1+n}} \langle B^* \mathbf{D}_x(\varphi_k v), B^* \mathbf{D}_x(\varphi_k v) \rangle \, d\nu \\ &= -\frac{1}{2} \int_{\mathbb{R}^{1+n}} \langle B^* \mathbf{D}_x v, B^* \mathbf{D}_x v \rangle \varphi_k^2 \, d\nu - \int_{\mathbb{R}^{1+n}} \langle B^* \mathbf{D}_x \varphi_k, B^* \mathbf{D}_x v \rangle v \varphi_k \, d\nu \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^{1+n}} \langle B^* \mathbf{D}_x \varphi_k, B^* \mathbf{D}_x \varphi_k \rangle v^2 \, d\nu \\ &\leq -\frac{1}{2} \|\varphi_k |B^* \mathbf{D}_x v|\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 + \frac{1}{4} \|\varphi_k |B^* \mathbf{D}_x v|\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 \\ &\quad + \frac{3}{2} \|v |B^* \mathbf{D}_x \varphi_k|\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2. \end{aligned}$$

Hence,

$$\begin{aligned} 0 &\geq \lambda \|\varphi_k v\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 + \frac{1}{4} \|\varphi_k |B^* \mathbf{D}_x v|\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 - \frac{3}{2} \|v |B^* \mathbf{D}_x \varphi_k|\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 \\ &\quad - C \|v\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 - \frac{1}{8} \|\varphi_k |B^* \mathbf{D}_x v|\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 - 2 \|v |B^* \mathbf{D}_x \varphi_k|\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 \\ &\geq \lambda \|\varphi_k v\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2 - C_1 \|v\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2. \end{aligned}$$

Letting $k \rightarrow \infty$, we obtain $0 \geq (\lambda - C_1) \|v\|_{L^2(\mathbb{R}^{1+n}, \nu)}^2$, which implies $v \equiv 0$ provided λ is large enough. \square

5.2. Proof of Theorem 1.2. The proof of Theorem 1.2 is the same of Theorem 1.1, with the space $E = H_{\#}^{0,4}(\mathbb{R}^{1+n}, \nu)$ instead of $H^{0,4}(\mathbb{R}^{1+n}, \nu)$ and using Lemma 3.5 instead of Lemma 3.4 in the first part, and functions φ_k depending only on x in the second part: $\varphi_k(x) = \varphi(x/k)$, with $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \equiv 1$ on $B(0, 1)$ and $\varphi \equiv 0$ outside $B(0, 2)$. We omit it.

The characterization of $D(G_{\#})$ given by Theorem 1.2 implies also that $D(G_{\#})$ is compactly embedded in $L_{\#}^2(\mathbb{R}^{1+n}, \nu)$, through the next Proposition.

Proposition 5.2. $H_{\#}^{1,2}(\mathbb{R}^{1+n}, \nu)$ is compactly embedded in $L_{\#}^2(\mathbb{R}^{1+n}, \nu)$.

Proof. Let $\mathcal{T}_{\#}$ be as in Lemma 4.4. Writing $H_{\#}^{1,2}(\mathbb{R}^{1+n}, \nu) = \mathcal{T}_{\#}^{-1} H_{\#}^{1,2}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})$, it suffices to show that $H_{\#}^{1,2}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})$ is compactly embedded in $L_{\#}^2(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})$.

Let $u \in B$, where B denotes the unit ball in $H_{\#}^{1,2}(\mathbb{R}^{1+n}, dt \times \mathcal{N}_{0,1})$. The logarithmic Sobolev inequality for the Gaussian measure $\mathcal{N}_{0,1}$ (see e.g. [Gro75, formula (1.2)]) yields, for each $t \in \mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{R}^n} |u(t, x)|^2 \log(|u(t, x)|) \mathcal{N}_{0,1}(dx) &\leq \int_{\mathbb{R}^n} |D_x u|^2 \mathcal{N}_{0,1}(dx) \\ &\quad + \|u(t, \cdot)\|_{L^2(\mathbb{R}^n, \mathcal{N}_{0,1})}^2 \log \|u(t, \cdot)\|_{L^2(\mathbb{R}^n, \mathcal{N}_{0,1})}. \end{aligned}$$

Hence, following e.g. the lines of [LMP06], for each $k > 1$ we obtain

$$\begin{aligned} \int_0^T \int_{B(0,R)^c} |u|^2 \mathcal{N}_{0,1}(dx) dt &\leq \int_0^T \int_{B(0,R)^c} \chi_E(x) k^2 \mathcal{N}_{0,1}(dx) dt \\ &\quad + \frac{1}{\log k} \int_0^T \int_{B(0,R)^c} \chi_{E^c}(x) |u|^2 \log |u| \mathcal{N}_{0,1}(dx) dt \\ &\leq k^2 T \mathcal{N}_{0,1}(B(0, R)^c) + \frac{T}{\log k}, \end{aligned}$$

where $E = \{|u| < k\}$. Therefore, given $\varepsilon > 0$, there exists $R > 0$, independent of u , such that

$$\int_0^T \int_{B(0,R)^c} |u|^2 \mathcal{N}_{0,1}(dx) dt \leq \varepsilon.$$

Since $L_{\#}^2((0, T) \times B(0, R), dt \times \mathcal{N}_{0,1}) = L_{\#}^2((0, T) \times B(0, R))$, and the embedding of $H_{\#}^{1,2}((0, T) \times B(0, R))$ into $L^2((0, T) \times B(0, R))$ is compact, we find $\{f_1, \dots, f_k\} \subset L^2((0, T) \times B(0, R))$ such that the balls $B(f_i, \varepsilon)$ cover the restrictions of the functions of B to $(0, T) \times B(0, R)$. Now, let \tilde{f}_i denote the extensions to $(0, T) \times \mathbb{R}^n$ by 0. Then $B \subset \cup_{i=1}^k B(\tilde{f}_i, 2\varepsilon)$ and the proof is complete. \square

6. PROOF OF THEOREM 1.3

Without loss of generality, we restrict ourselves to the case $T_1 = a < 0$ and $T_2 = 0$. We first consider the problem

$$\begin{cases} u_s(s, x) + \mathcal{L}(s)u(s, x) = 0, & s \in (a, 0), \ x \in \mathbb{R}^n, \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^n. \end{cases}$$

Proposition 6.1. *For each $\varphi \in H^1(\mathbb{R}^n, \nu_0)$ the function $(s, x) \mapsto u(s, x) := (P_{s,0}\varphi)(x)$ belongs to $H^{1,2}((a, 0) \times \mathbb{R}^n, \nu)$ and there exists $C > 0$, independent of φ , such that*

$$(6.1) \quad \|u\|_{H^{1,2}((a,0) \times \mathbb{R}^n, \nu)} \leq C \|\varphi\|_{H^1(\mathbb{R}^n, \nu_0)}.$$

Proof. We use the following identities:

$$(6.2) \quad [\mathcal{L}(s), D]P_{s,0}\varphi = (\mathcal{L}(s)D - DL(s))P_{s,0}\varphi = A(s)DP_{s,0}\varphi = A(s)U^*(s, 0)P_{s,0}D\varphi,$$

and, for $\psi \in H^3(\mathbb{R}^n, \nu_s)$,

$$(6.3) \quad \int_{\mathbb{R}^n} |D\psi|^2 \partial_s \rho(s, x) dx = 2 \int_{\mathbb{R}^n} \langle \mathcal{L}(s)D\psi, D\psi \rangle d\nu_s + \int_{\mathbb{R}^n} |B^*(s)D^2\psi|^2 d\nu_s.$$

Formula (6.2) follows from the explicit expressions of $\mathcal{L}(s)$ and $P_{s,0}$, while (6.3) follows from Lemma 2.4 and the identity $\mathcal{L}(s)(\varphi^2) = 2\varphi\mathcal{L}(s)\varphi + |B^*(s)D\varphi|^2$, applied to each derivative $D_j\psi$.

Thus, we obtain

$$\begin{aligned} \partial_s \int_{\mathbb{R}^n} |DP_{s,0}\varphi|^2 \nu_s(dx) &= -2 \int_{\mathbb{R}^n} \langle D\mathcal{L}(s)P_{s,0}\varphi, DP_{s,0}\varphi \rangle \nu_s(dx) + \int_{\mathbb{R}^n} |DP_{s,0}\varphi|^2 \partial_s \rho(s, x) dx \\ &= -2 \int_{\mathbb{R}^n} \langle \mathcal{L}(s)DP_{s,0}\varphi, DP_{s,0}\varphi \rangle \nu_s(dx) \\ &\quad + 2 \int_{\mathbb{R}^n} \langle [\mathcal{L}(s), D]P_{s,0}\varphi, DP_{s,0}\varphi \rangle \nu_s(dx) + \int_{\mathbb{R}^n} |DP_{s,0}\varphi|^2 \partial_s \rho(s, x) dx \\ &= \int_{\mathbb{R}^n} |B^*D^2P_{s,0}\varphi|^2 \nu_s(dx) \\ &\quad + 2 \int_{\mathbb{R}^n} \langle A(s)U^*(s, 0)P_{s,0}D\varphi, U^*(s, 0)P_{s,0}D\varphi \rangle \nu_s(dx). \end{aligned}$$

Integrating with respect to s , we obtain

$$\begin{aligned} \| |D\varphi| \|_{L^2(\mathbb{R}^n, \nu_0)}^2 - \| |DP_{a,0}\varphi| \|_{L^2(\mathbb{R}^n, \nu_a)}^2 &= \int_a^0 \int_{\mathbb{R}^n} |DP_{s,0}\varphi|^2 \nu_s(dx) ds \\ &= \int_a^0 \int_{\mathbb{R}^n} (|B^*(s)D^2P_{s,0}\varphi|^2 + 2\langle A(s)U^*(s, 0)P_{s,0}D\varphi, U^*(s, 0)P_{s,0}D\varphi \rangle) \nu_s(dx) ds. \end{aligned}$$

Since $\|B^*(s)^{-1}\| \leq 1/\mu_0$ by assumption (1.8), then

$$\| |D_x^2 u| \|_{L^2((a,0) \times \mathbb{R}^n, \nu)} \leq \frac{1}{\mu_0^2} \int_a^0 \int_{\mathbb{R}^n} |B^*(s) D^2 P_{s,0} \varphi|^2 \nu_s(dx) ds$$

and, hence,

$$\| |D_x^2 u| \|_{L^2((a,0) \times \mathbb{R}^n, \nu)} + \| |D_x u(a, \cdot)| \|_{L^2(\mathbb{R}^n, \nu_a)} \leq C(T) \| |D\varphi| \|_{H^1(\mathbb{R}^n, \nu_0)},$$

where $C(a) > 0$ is independent of φ . Since $\partial_s u = -\mathcal{L}(s)u(s, \cdot)$ the statement follows using estimate (4.1). \square

We also need the following lemma about the traces at $t = 0$ of functions belonging to $H^{1,2}((a, 0) \times \mathbb{R}^n, \nu)$.

Lemma 6.2. *We have*

$$H^1(\mathbb{R}^n, \nu_0) = \{u(0, \cdot) : u \in H^{1,2}((a, 0) \times \mathbb{R}^n, \nu)\},$$

and the norm

$$\varphi \mapsto \inf\{\|u\|_{H^{1,2}((a,0) \times \mathbb{R}^n, \nu)} : u(0, \cdot) = \varphi\}$$

is equivalent to the norm of $H^1(\mathbb{R}^n, \nu_0)$.

Proof. By Lemma 4.3, we have $\mathcal{T}_0 u \in H^{1,2}((a, 0) \times \mathbb{R}^n, \nu_0)$ and there exists $C > 0$, independent of u , such that

$$\|\mathcal{T}_0 u\|_{H^{1,2}((a,0) \times \mathbb{R}^n, \nu_0)} \leq C \|u\|_{H^{1,2}((a,0) \times \mathbb{R}^n, \nu)}.$$

Therefore, by standard arguments,

$$\|u(0, \cdot)\|_{H^1(\mathbb{R}^n)} = \|(\mathcal{T}_0 u)(0, \cdot)\|_{H^1(\mathbb{R}^n, \nu_0)} \leq C \|\mathcal{T}_0 u\|_{H^{1,2}((a,0) \times \mathbb{R}^n, \nu_0)} \leq C \|u\|_{H^{1,2}((a,0) \times \mathbb{R}^n, \nu)},$$

where C is independent of u . On the other hand, Proposition 6.1 states that for each $\varphi \in H^1(\mathbb{R}^n, \nu_0)$ the function $u(s, x) = (P_{s,0} \varphi)(x)$ belongs to $H^{1,2}((a, 0) \times \mathbb{R}^n, \nu)$, with estimate (6.1). The statement follows. \square

Finally, we are in the position to prove Theorem 1.3. Let $f \in L^2((a, 0) \times \mathbb{R}^n, \nu)$, fix $\lambda > 0$ and set

$$f_\lambda(s, x) = \begin{cases} -e^{\lambda s} f(s, x) & x \in \mathbb{R}^n, s \in (a, 0), \\ 0 & x \in \mathbb{R}^n, s \notin (a, 0) \end{cases}$$

Then $f_\lambda \in L^2(\mathbb{R}^{1+n}, \nu)$ and, by Theorem 1.1, $u_\lambda := (\lambda - G)^{-1} f_\lambda \in H^{1,2}(\mathbb{R}^{1+n}, \nu)$, and $\|u_\lambda\|_{H^{1,2}(\mathbb{R}^{1+n}, \nu)} \leq C \|f_\lambda\|_{L^2(\mathbb{R}^{1+n}, \nu)}$ with C independent on f . Moreover, $u_1(s, x) := e^{-\lambda s} u_\lambda(s, x)$, $x \in \mathbb{R}^n$, $s \in (a, 0)$ satisfies

$$\partial_s u_1(s, x) + \mathcal{L}(s)u_1(s, x) = -\lambda e^{-\lambda s} u_\lambda(s, x) + e^{-\lambda s} \mathcal{G}u_\lambda = -e^{-\lambda s} f_\lambda(s, x) = f(s, x)$$

for $x \in \mathbb{R}^n$ and $s \in (a, 0)$. Furthermore, there exists $C_1 > 0$, independent of f , such that

$$\|u_1\|_{H^{1,2}((a,0) \times \mathbb{R}^n)} \leq C_1 \|f\|_{L^2((a,0) \times \mathbb{R}^n)}.$$

Hence, by Lemma 6.2,

$$u := u_1 + P_{\cdot,0}(u_0 - u_1(0, \cdot)) \in H^{1,2}((a, 0) \times \mathbb{R}^n, \nu)$$

and u satisfies (1.17) and (1.18).

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